Algebraic and Spectral Graph Theory | Oct 21, 2023

Electrical Networks and Graphs

Columbia Undergraduate Learning Seminar in TCS | Presenter: *Ryan Anselm*

1 Electrical Networks

We can make an observation that *resistor networks* are a useful analogy to undirected weighted graphs $G = (V, E, w)$. If an edge (i, j) has weight w_{ij}, we give the corresponding resistor in a network resistance

$$R_{ij} = 1/w_{ij}.$$

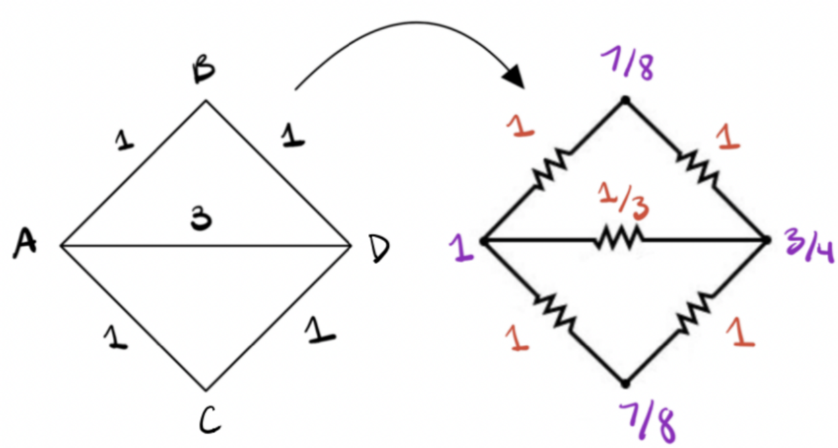
For each edge (i, j) we will also define the current flowing from i to j as I_{ij}. Because current is a directed quantity we have that

$$I_{ij} = -I_{ji}.$$

Each node will also be at some voltage. We specify the voltage at node i as v_i. Current flows from high voltage to low voltage following our version of Ohm's law:

$$I_{ij} = \frac{v_i - v_j}{R_{ij}} = w_{ij}(v_i - v_j)$$

Suppose that we make current flow through the network by feeding current into some nodes and drawing it out of other nodes according to an external curent vector $I_{ext} \in \mathbb{R}^n$, where $I_{ext}(a) > 0$ if there is external current flowing into a and $I_{ext}(a) < 0$ if there is external current flowing out of a. This external current vector will induce voltages $\vec{v}$ on all of the nodes and currents on all of the edges (Assume that $I_{ext} \perp \vec{1} \iff \sum_a I_{ext}(a) = 0$ so there is no net accumulation of charge).



The Laplacian of the graph above is

$$L = \begin{bmatrix} 5 & -1 & -1 & -3 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -3 & -1 & -1 & 5 \end{bmatrix}$$

Imagine first that we are given the vector of induced voltages v on all of the nodes caused by some external current, and we wish to find what the external current is. Using Ohm's law, the net current *coming* from node a is

$$I_a = \sum_{(a,b)\in E} I_{ab} = \sum_b w_{ab}(v_a - v_b) = (Lv)_a.$$

Hence,

$$I_{ext} = Lv.$$

Suppose we are instead given vector of external currents I_{ext} and wish to find the induced voltages on all of the nodes.

Definition 1. *The pseudoinverse of the Laplacian L^+ is the unique matrix satisfying:*

1. $L^+\vec{1} = 0$
2. *For all* $w \perp \vec{1} : L^+ w = v$ *s.t.* $Lv = w$ *and* $v \perp \vec{1}$

Observe that $\forall v \perp \vec{1}$,

$$L^+ I_{ext} = L^+(Lv) = v$$

The pseudoinverse of the Laplacian, L^+, is an object of interest if we want to find the induced voltages on some circuit given just the external currents on that circuit.

2 Effective Resistance

The *effective resistance* is the resistance we would get if we view the entire network as one complex resistor between two nodes. Recall the rules for computing effective resistance from physics.

- In Series: $R_{tot} = R_1 + R_2$
- In Parallel: $R_{tot} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$

Suppose that $I_{ext} = e_a - e_b$ where e_a is the vector that is 1 on index a and 0 everywhere else. If we viewed the entire circuit as a giant resistor with some external currents coming in and out, the current passing through the circuit as a whole would just be 1. The resulting voltages are

$$v = L^+(e_a - e_b)$$

The voltage difference between nodes c and d in this graph would be

$$v_c - v_d = (e_c - e_d)^T v = (e_c - e_d)^T L^+(e_a - e_b)$$

If we set $c = a$ and $d = b$, we are now finding the voltage difference on the same nodes that were the source and sink. Then, $v_a - v_b = (e_a - e_b)^T L^+ (e_a - e_b) = I_{ab} R_{eff}(a, b)$ If we view the network as a complex resistor between a and b, the current passing through it is $I_{ab} = 1$, so we have

$$\begin{aligned} R_{eff}(a, b) &= (e_a - e_b)^T L^+ (e_a - e_b) \\ &= (e_a - e_b)^T L^{+/2} L^{+/2} (e_a - e_b) \\ &= ||L^{+/2}(e_a - e_b)||^2 \\ &= ||L^{+/2} e_a - L^{+/2} e_b||^2 \end{aligned}$$

where $L^{+/2}$ is the "square root matrix" of L^+ such that $L^{+/2} L^{+/2} = L^+$. This happens to always exist when L^+ is positive semidefinite.

Theorem 2. *R_{eff} is a metric on the nodes of the graph.*

1. *$R_{eff}(a, b) \geq 0$ for all a, b, and $R_{eff}(a, a) = 0$.*
2. *$R_{eff}(a, b) = R_{eff}(b, a)$ for all a, b*
3. *Triangle Inequality: $R_{eff}(a, c) \leq R_{eff}(a, b) + R_{eff}(b, c)$ for all a, b, c.*

Proof. The first two conditions are trivial to show. For the triangle inequality, it suffices to consider graphs with three vertices because we can reduce any graph to one on just vertices a, b, c without changing the effective resistances between them. Let $z = w_{ab}$, $y = w_{ac}$, $x = w_{bc}$. If we eliminate vertex c, we create an edge between a and b of weight

$$\frac{xy}{x + y}$$

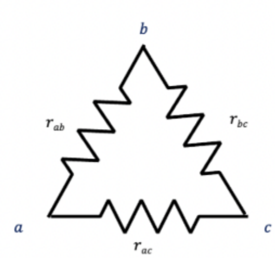
Then by adding this to the existing edge, we get

$$R_{eff}(a, b) = \frac{1}{z + \frac{xy}{x+y}} = \frac{x + y}{zx + zy + xy}.$$

Then we just show that for all positive x, y, z that

$$\frac{x + y}{zx + zy + xy} + \frac{y + z}{zx + zy + xy} \geq \frac{x + z}{zx + zy + xy}$$

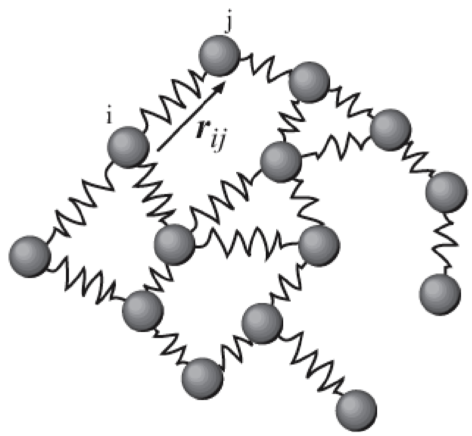
□



Theorem 3. *(Rayleigh's Monotonicity Principle) If we decrease resistances (i.e. increase edge weights), the effective resistances cannot increase.*

3 Spring Networks

Let's turn our attention to another physically motivated model. Think about a graph $G = (V, E)$ where the nodes V are tennis balls and the edges E are springs between the balls. Hooke's Law says that the spring between balls i and j exerts a force $\vec{F}_{ij} = -k_{ij}(\vec{x}_i - \vec{x}_j)$ where $\vec{x}_i$ and $\vec{x}_j$ are the positions of balls i and j for $\vec{x}_i, \vec{x}_j \in \mathbb{R}^d$ and k_{ij} is the spring constant, representing the "stiffness" of the spring between i and j.



Imagine that some of the balls $B \subseteq V$ are nailed down by us in specific positions (otherwise, all the balls would just collapse into a big pile). Physical intuition tells us that there must be *some* stable configuration of the unfixed balls it eventually reaches. For a stable configuration to exist, all of the balls that are not nailed down must have all the forces acting on them from all of the attached springs balance out. Let $S = V - B$ be the set of untaped balls. Newton's second law requires that for every $j \in S$

$$\sum_{(i,j)\in E} F_{ij} = \sum_{j:(i,j)\in E} -k_{ij}(x_i - x_j) = 0$$

$$\iff \deg(j) \cdot x_j = \sum_{(i,j)\in E} k_{ij} x_i$$

$$\iff x_j = \frac{1}{\deg(j)} \sum_{(i,j)\in E} w_{ij} x_i$$

where the weighted degree $\deg(j) := \sum_{(i,j)\in E} k_{ij}$. Notice that this implies that the position of ball j is the **weighted average** of all of its neighbors. We call the function $x : V \to \mathbb{R}^d$ that maps balls to their equilibrium positions a *harmonic function* because of this condition that each ball is in a position that is the weighted average of its neighbors. This is again an analogy to harmonic functions f over continuous domains which satisfy the condition $\nabla^2 f = 0$.

Harmonic function property	Discrete	Continuous
Mean value theorem	$x_j = \frac{1}{\deg(j)} \sum_{(i,j)\in E} w_{ij} x_i$	$f(x) = \int_{B_r(x)} f dx$
Uniqueness of solution	Yes	Yes

Another way of thinking about it is the energy perspective. The spring between balls i and j contributes potential energy $U_{ij} = \frac{1}{2}k_{ij}||x_i - x_j||^2$. A real system of springs will try to relax into a minimum energy configuration. In other words, a real system tries to reach an energy of

$$U_{min} = \min \sum_{(i,j)\in E} U_{ij} = \min_{\mathbf{x}} \frac{1}{2} \sum_{(i,j)\in E} k_{ij}(x_i - x_j)^2$$

by finding an optimal configuration of balls. For any x that minimizes the energy, the gradient of the energy with respect to x must be 0. The partial derivative is

$$\sum_{(i,j)\in E} k_{ij}(x_i - x_j) = 0$$

Recall that

$$x^T L x = \sum_{(i,j)\in E} w_{ij}(x_i - x_j)^2.$$

If we define the weights of the graph such that $w_{ij} = \frac{1}{2}k_{ij}$, then the minimum energy configuration is $\arg\min x^T L x$.

References

[1] Salil Vadhan, CS 229cr Spectral Graph Theory in Computer Science, Lecture 16, https://drive.google.com/file/d/1DwgPcqPRVoA4bdNtFu6iRB6oUIzkiKYH/view

[2] Daniel Spielman, Spectral and Algebraic Graph Theory Ch. 11-12, https://cs-www.cs.yale.edu/homes/spielman/sagt/sagt.pdf