Sample Spanning Trees

Algebraic and Spectral Graph Theory

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Preliminaries

Sometimes, you are given a graph $G = (V, E)$, but you only want one way to connect its nodes. We can derive a subgraph of G called a spanning tree.

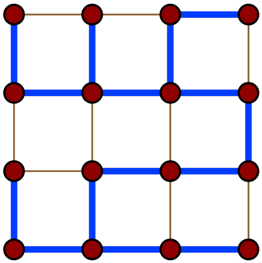

Figure: A spanning tree of a 4x4 grid graph.

The Laplacian Matrix

Recall the matrix **L** from previous lectures whereby

$$D_{ij} = \begin{cases} \text{deg}(v_i), i = j \\ 0, otherwise \end{cases} \quad \Rightarrow L = D - A$$

Where A is the adjacency matrix of the graph. We can think of the Laplacian as a measure of "smoothness" over the graph (recall local variance from earlier lectures to highlight differences in smoothness).

Determinants

There are several ways to compute the determinants of $n \times n$ matrices, but for now we'll focus on the Leibniz Formula which expresses the elements of a square matrix in terms of the permutations of the matrix elements (using permutation groups from abstract algebra). Suppose **A** is a $n \times n$ square matrix, where $\mathbf{A}(i, j)$ is an entry (this is the notation Spelman uses), then the determinant of **A** is

$$\det(\mathbf{A}) = \sum_{\pi \in S_n} \left(\text{sgn}(\pi) \prod_{i=1}^{n} \mathsf{A}(i, \pi(i))\right)$$

This formula isn't particularly useful for actually computing determinants. It's used here for a better theoretic understanding.

Characteristic Polynomials

From any classic linear algebra textbook, the characteristic polynomial of a matrix **A** is $\det(x\mathbf{I} - \mathbf{A})$. You can compute the polynomial by using any determinant formula. For our purposes, the characteristic polynomial is defined as:

$$\sum_{k=0}^{n} x^{n-k}(-1)^k \sigma_k(A)$$

In which $\sigma_k(A)$ is the *k*th elementary symmetric function (an invariant polynomial) of the eigenvalues of **A**, counted with algebraic multiplicity.

$$\sigma_k(A) = \sum_{|S|=k} \prod_{i \in S} \lambda_i$$

Where, S is the set of indexes $\{1, 2, ..., n\}$.

Further notes about determinants and characteristic polynomials

Elementary symmetric functions e_n are defined as follows without loss of generality:

$$\begin{aligned} e_1(x_1, x_2.x_3) &= x_1 + x_2 + x_3 \\ e_2(x_1, x_2.x_3) &= x_1x_2 + x_1x_3 + x_2x_3 \\ e_3(x_1, x_2, x_3) &= x_1x_2x_3 \end{aligned} \tag{1}$$

and so on and so forth. These elementary symmetric functions denote the k-wise products of eigenvalues of A. As such, $\sigma_1(A)$ is its trace and $\sigma_n(A)$ is its determinant (by Leibniz). Also recall that determinants are multiplicative: $\det(A)\det(B) = \det(AB)$.

What are we building towards?

Spelman's SAGT book as well as a lot of papers on spectral graph theory assume a great deal of mathematical background. Heck, just trying to interpret Leibniz's determinant formula requires knowledge of the theory of symmetric groups and some functional analysis on top of the already mathematically dense linear algebra. The purpose of all the preliminaries is to build up towards to the central result in SAGT that relates determinants with the number of spanning trees of a given graph.

Kirchoff's Matrix-Tree Theorem

Theorem (Kirchoff). Let $G = (V, E, w)$ be a connected, weighted graph. Then,

$$\sigma_{n-1}(\mathbf{L}_G) = n \sum_T \prod_{e \in T} w_e$$

This is a modified version of the standard theorem which I will discuss later. Let's focusing on what's actually happening here.

1. L_G is the Laplacian Matrix defined earlier of the graph G.
2. σ_{n-1} is the symmetric $n-1$th elementary function of the eigenvalues of the Laplacian Matrix.
3. The theorem essentially states that the function is equal to n times the sum over the product of the total weight of each spanning tree.

Proof.

Lemma. Let $G = (V, E, w)$ be a weighted **tree**.

$$\sigma_{n-1}(\mathbf{L}_G) = n \prod_{e \in E} w_e$$

From the textbook, we know $\sigma_{n-1}(\mathbf{L}_G) = \sum_{a \in V} \det(\mathbf{L}_G(S_a, S_a))$. In simpler terms, the $(n-1)$th elementary symmetric functions of all the eigenvalues of $\mathbf{L}_G$ is the sum of the determinants of the submatrices of the Laplacian for each vertex (very similar to the Cofactor Expansion). The rest of the proof is quite complicated and long (pg. 112-113 of SAGT).

Proof.

However, the general gist of it is:

- $\mathbf{L}_G = \mathbf{U}^{\mathsf{T}}\mathbf{W}\mathbf{U}$ and $\mathbf{B} = \mathbf{W}^{\frac{1}{2}}\mathbf{U}$.
- $\mathbf{U}$ is the signed edge-vertex adjacency matrix and $\mathbf{W}$ is the diagonal matrix of edge weights.
- We show that $\sigma_{n-1}(\mathbf{L}_G) = \sigma_{n-1}(\mathbf{B}^{\mathsf{T}}\mathbf{B})$ implies that $\sigma_{n-1}(\mathbf{L}_G) = \sum_{|S|=n-1, S\subset E} \sigma_{n-1}(\mathbf{L}_{G_S})$.

We are basically showing that the sum of the determinants of all principal cofactors of $\mathbf{L}_G$ can be expressed in terms of $\mathbf{B}$ and $\mathbf{B}^T$, decomposing the function into a sum of determinants of their submatrices.

Understanding

It should be noted that Spelman's proof in the book places a heavy reliance on dense notation. The proof of this theorem, in my opinion, is stated in a much cleaner way on Wikipedia (Article No. "Kirchoff's Theorem"). This proof utilizes the identity proven in the Cauchy-Binet formula which you might've seen in linear algebra. It's essentially the same logic as Spelman's but easier to comprehend.

Kirchoff's Theorem Simplified

What if we just want to count the number of spanning trees in a given connected graph? We can use a more simplified version of the Matrix Three Theorem. For a given connected graph G with n vertices, let $\lambda_1, \lambda_2, ..., \lambda_{n-1}$ be the non-zero eigenvalues of its Laplacian matrix. Then the number of spanning trees is

$$t(G) = \frac{1}{n}\lambda_1, \lambda_2, ..., \lambda_{n-1}$$

This is just the previous theorem, but assuming the graph is unweighted. We divide by n because we are essentially taking the cofactor determinant n times in the initial summation.

Example (from Wikipedia)

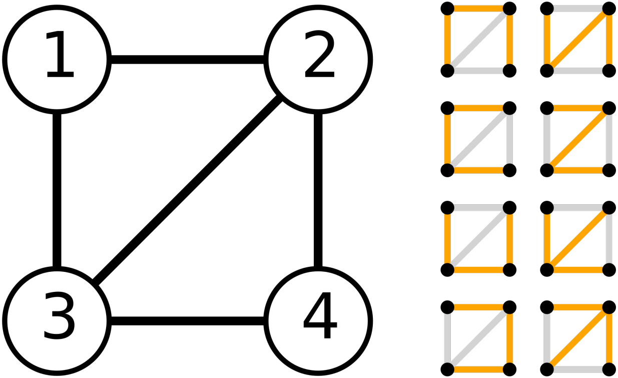

Figure: A diamond graph **G** and its corresponding trees.

Example (from Wikipedia)

The Laplacian Matrix **L** for this graph is

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

Take any cofactor determinant: $C_{1,1} = 8$. This is the number of spanning trees of G. Every cofactor determinant will evaluate to 8.

Leverage Scores

This is the last topic. The leverage score of a given edge is written as l_e and is defined to be $w_e R_{eff}(e)$ or the weight of the edge times the effective resistence between its endpoints. Recall from last time that

$$R_{eff} = (\delta_a - \delta_b)^\mathsf{T} L_G^{+} (\delta_a - \delta_b)$$

Details can be seen on last week's notes of what this formally means. If we choose a spanning tree T with a probability proportional to the product of its edge weights, then for every edge e

$$Pr[e \in T] = l_e$$

The probability that e is an edge in T is exactly its leverage score.