Cheeger's Inequality and Expander Graphs

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A brief overview

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- We can formalize what it means for graphs to be expanders in a few ways; one such way is using the graph's conductance $\phi(G)$

A brief overview

- Expanders are a special type of graph we'd like to characterize. To do this, we'll give qualitative examples and then define a notion of expansion
- We can formalize what it means for graphs to be expanders in a few ways; one such way is using the graph's conductance $\phi(G)$
- Cheeger's inequality connects the second smallest eigenvalue of the graph Laplacian with $\phi(G)$! We'll build up to it:

$$\frac{\lambda_2(N_G)}{2} \leq \phi(G) \leq \sqrt{2\lambda_2(N_G)}$$

Expanders, qualitatively

An expander graph is a graph that is

1. Sparse: Has a small number of edges
2. Well-connected: Many paths from one node to another
   - Every subset of vertices that is not too large has a large boundary
   - Need to cut a lot of edges to 'cut' graph into 2 disconnected parts

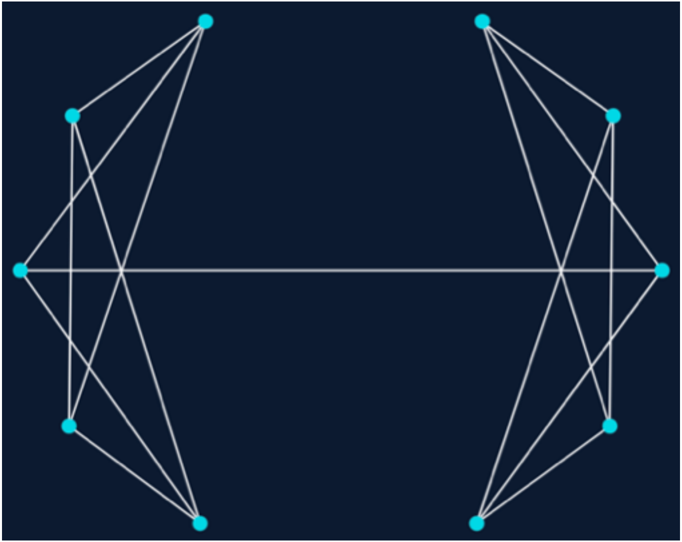


Figure: Not an expander; 1) has few edges satisfied, but 2) not well-connected, since cutting middle edge breaks it in half

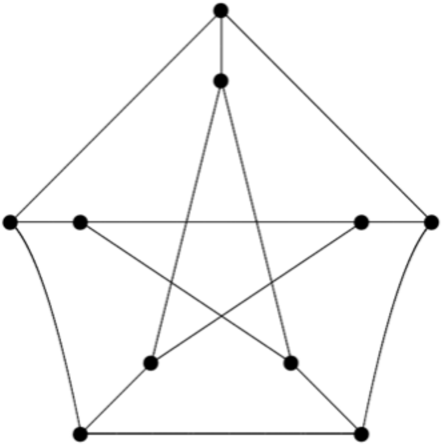


Figure: Petersen graph (type of expander called Ramanujan graph)

What we need to know

- To make sense of Cheeger's inequality, we need to briefly recall the Laplacian L and normalized Laplacian N
- Then we will state the 2 characterizations of expanders connected by Cheeger's inequality:
  1. Spectral: Using the kth eigenvalue of L and N
  2. Vertex: Using conductance

Conductance is one way to formalize the idea of expanders

Let $G = (V, E)$ be an undirected, unweighted graph where $S \subseteq V$ is some set of nodes

<u>Def</u> (Conductance of S) $\phi(S) := \frac{w(\partial(S))}{\min\{d(S), d(V \setminus S)\}}$ where

- $\partial(S)$ is the boundary of S (edges, the ones 'touching' S)
- $w(\partial(S))$ is the sum of the edges' weights
- $d(S)$ is the sum of degrees of vertices in S

$\rightarrow$ Basically, measures how connected S is, scaled down by the degrees of nodes in S. This looks a lot like the 2 criteria!

One interpretation: $\phi(S) =$ probability of escaping S in a random walk

Conductance is one way to formalize the idea of expanders

Conductance of a graph is just the minimum conductance amongst all $S \subset V$:

$$\phi(G) = \min_{S \subset V} \phi(S)$$

In the same vein, can interpret $\phi(G)$ as the probability of escaping the most poorly connected set in a random walk

Computing conductance is NP-hard

Example: $\phi(G) > 0$ iff G is connected, bigger $\phi(G)$ means more well-connected, lower edge degree

Conductance is one way to formalize the idea of expanders

Now we can define expanders:

Say that G is a φ-expander iff $\phi(G) \geq \varphi$

Now ready to connect this to the graph Laplacian!

Recall the normalized Laplacian N

- Given a graph, enumerate the vertices $\{1, \ldots, n\}$
- Its graph Laplacian is $L = D - A$, where
  - D is the diffusion matrix
  - A is the adjacency matrix
- L is a real symmetric matrix, so by the spectral theorem it has n real eigenvalues
- Also useful to construct the normalized Laplacian N, which scales L down by degree in a way that preserves symmetry, $N = D^{-1/2} L D^{-1/2}$

Eigenvalues of N, specifically λ_2

- Recall the enumeration of eigenvalues from lecture 2, where λ_k is the kth smallest eigenvalue
- Eigenvalues of N:
  - $\lambda_1 = 0$ and $\lambda_n \leq 2$
  - $\lambda_k = 0$ iff G has at least k connected components (means that $\lambda_1 = 0$)
  - Also means that λ_2 (also called the Fiedler value/algebraic connectivity of G) is > 0 iff the graph connected
- We can compute λ_2 efficiently!

Seems like λ_2 can characterize expander properties

Here is Cheeger's inequality again:

$$\frac{\lambda_2(N_G)}{2} \leq \phi(G) \leq \sqrt{2\lambda_2(N_G)}$$

Kind of know what it means now:
If G is an φ-expander, we have a bound on $\lambda_2(N_G)$

$\rightarrow$ To test whether G is an φ-expander, we can compute $\lambda_2(N_G)$ to get a range for $\phi(G)$, which is good because computing $\phi(G)$ is NP-hard but computing $\lambda_2(N_G)$ isn't

$\rightarrow$ Big $\lambda_2(N_G)$ means more expander-like

Overview of Cheeger's inequality proof

Direction 1: $\frac{\lambda_2(N_G)}{2} \leq \phi(G)$

(Low conductance $\implies$ small eigenvalue)

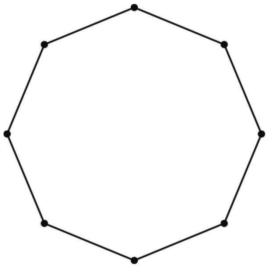
- Let $S \subseteq V$ be the subset where $\phi(S) = \phi(G)$
- For every S, its $\phi(S) =$ Rayleigh quotient of $\mathbf{1}_S$ whose v-th coordinate is 1 iff $v \in S$, $R_L(\mathbf{1}_S)$, so $R_L(\mathbf{1}_S) \leq \phi(G)$
- Variational characterization of eigenvalues tells us that the second eigenvalue is the minimum value of $\frac{x^T N_G x}{x^T x}$ for all $x \perp v_1$, the first eigenvector
- This means that $\lambda_2 \leq 2\phi(G)$ as desired if all vectors in the 2-dimensional space X of linear combinations of $\mathbf{1}_S$, $\mathbf{1}_{V \setminus S}$ have Rayleigh quotients at most $2\phi(G)$
- This will follow from N_G being positive semidefinite

Overview of Cheeger's inequality proof

Direction 2: $\phi(G) \leq \sqrt{2\lambda_2}$
(Small eigenvalue $\implies$ low conductance)

- Connect conductance to sparse cuts (mentioned in previous lecture) to show that given any $x \perp v_1$, we can find a cut S in G s.t. $\phi(S) \leq \sqrt{2\frac{x^T L_G x}{x^T D x}}$
- Fiedler's algorithm gives a way to find such an S

Examples that show Cheeger's inequality is tight



Consider the cycle with $4n$ vertices. Then $\phi(G) \leq \sqrt{2\lambda_2}$ is tight up to a constant, since λ_2 and $\phi(G)$ are both $\Theta(1/n^2)$

Examples that show Cheeger's inequality is tight

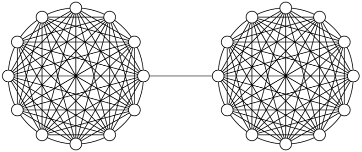


Figure: Dumbbell graphs are complete graphs connected by a single edge, similar to the example shown in the beginning

Consider the dumbbell graph with $2n$ vertices. Using the Courant-Fischer characterization for λ_2 from the 2nd seminar, we can show that $\lambda_2 \geq c/n^2$ for some $c > 0$. Using random walk properties, we can show that $\phi(G)$ is $\Theta(1/n^2)$ so $\frac{\lambda_2(N_G)}{2} \leq \phi(G)$ is tight